1 Quiz 1

Problem 1.1 (1pt)**.** Consider first order equations and determine if they are linear homogeneous, linear inhomogeneous, quasilinear or non-linear (u is an unknown function):

$$u_t + xu_x = 0, \qquad \underline{\hspace{4cm}} \tag{1.1}$$

$$u_t + uu_x = 0. \qquad \underline{\hspace{4cm}} \tag{1.2}$$

Answer. (7.1) Linear homogeneous, (7.2) Quasilinear □

Problem 1.2 (2pt)**.** Find the general solutions to the following equation:

$$u_{xxyy} = \sin(x)\sin(y) \tag{1.3}$$

Solution. Integrating by x, x, y and y:

$$\begin{aligned} u_{xxyy} = \sin(x)\sin(y) \implies u_{xxy} &= -\sin(x)\cos(y) + \phi''(x) \implies \\ u_{xx} &= -\sin(x)\sin(y) + \phi''(x)y + \phi_1''(x) \implies \\ u_x &= \cos(x)\sin(y) + \phi'(x)y + \phi_1'(x) + \psi(y) \implies \\ u &= \sin(x)\sin(y) + \phi(x)y + \phi_1(x) + \psi(y)x + \psi_1(y) \end{aligned}$$

where $\phi, \phi_1, \psi, \psi_1$ are arbitrary functions. □

Problem 1.3 (2pt)**.** Find the solution of

$$\begin{cases} u_x + 3u_y = xy, \\ u|_{x=0} = 0. \end{cases} \tag{1.4}$$

Solution.

$$\frac{dx}{1} = \frac{dy}{3} = \frac{du}{3xy} \implies y - 3x = C_1 \implies du = x(3x + C_1)dx \implies$$
$$u = x^3 + \frac{1}{2}C_1x^2 + C_2$$

with $C_2 = \phi(C_1)$ with arbitrary function ϕ; so

$$u = x^3 + \frac{1}{2}(y-3x)x^2 + \phi(y-3x) = \frac{1}{2}x^2y - \frac{1}{2}x^3 + \phi(y-3x)$$

is the general solution to equation. Plugging into initial condition we get $\phi(y) = 0$ and thus

$$u = x^3 + \frac{1}{2}(y-3x)x^2 + \phi(y-3x) = \frac{1}{2}x^2y - \frac{1}{2}x^3$$

is the final answer. □

2 Quiz 2

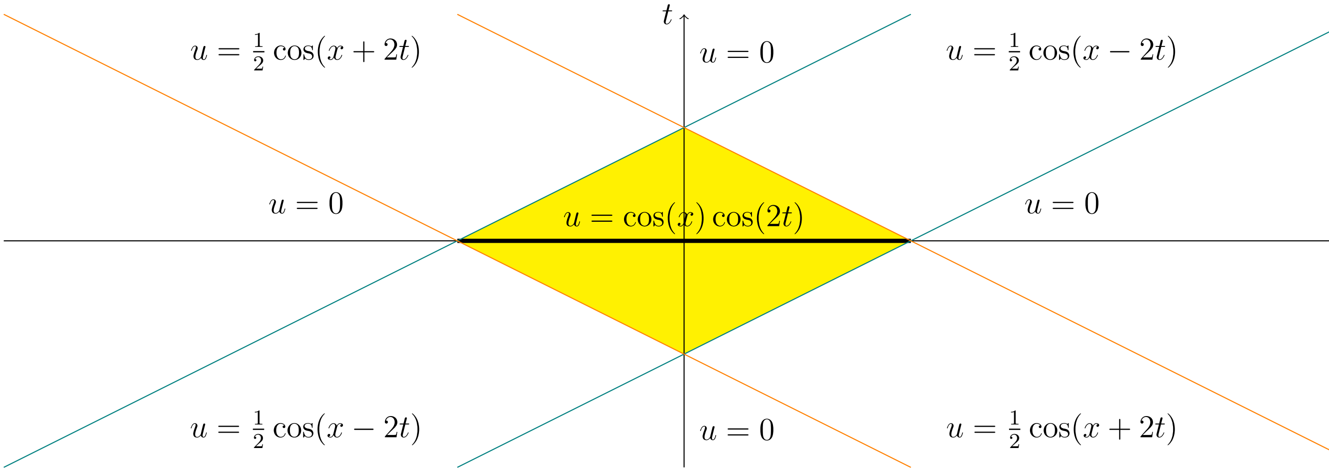
Problem 2.1 (5pt)**.** Solve IVP

$$u_{tt} - 4u_{xx} = 0, \tag{2.1}$$
$$u|_{t=0} = g(x), \quad u_t|_{t=0} = h(x) \tag{2.2}$$

with

$$g(x) = \begin{cases} \cos(x) & |x| < \pi/2, \\ \\ 0 & |x| \ge \pi/2, \end{cases} \qquad h(x) = 0. \tag{2.3}$$

Solution. See figure; here $(-\frac{1}{2}, \frac{\pi}{2})$ (on $t = 0$) is bold, characteristics passing through its ends $x - 2t = \text{const}$ and $x + 2t = \text{const}$ are teal and orange and in the central diamond $u = \frac{1}{2}\cos(x + 2t) + \frac{1}{2}\cos(x - 2t) = \cos(x)\cos(2t)$.



$\square$

3 Quiz 3

Problem 3.1 (5pt)**.** Find solution

$$\begin{aligned}
&u_{tt} - c_1^2 u_{xx} = 0, && && t>0, x>0, && (3.1)\\
&u_{tt} - c_2^2 u_{xx} = 0, && && t>0, x<0, && (3.2)\\
&u|_{t=0} = \phi(x), && u_t|_{t=0} = c_1\phi'(x) && x>0, && (3.3)\\
&u|_{t=0} = 0, && u_t|_{t=0} = 0, && x<0, && (3.4)\\
&u|_{x=+0} = \alpha u|_{x=-0}, && u_x|_{x=+0} = \beta u_x|_{x=-0} && t>0 && (3.5)
\end{aligned}$$

(separately in $x > c_1t$, $0 < x < c_1t$, $-c_2t < x < 0$ and $x < -c_2t$.

Solution. See figure. Observe that in $\{x > 0\}$, $\{x < -c_2t\}$ the general solution is

$$u(x,t) = \varphi_1(x + c_1t) + \psi_1(x - c_1t), \tag{3.6}$$
$$u(x,t) = \varphi_1(x + c_2t) + \psi_1(x - c_2t), \tag{3.7}$$

due to (7.1), (7.2) respectively. Plugging (6.6) and (4.7) into (6.3) and (6.4) respectively we get

$$\begin{aligned}
&\varphi_1(x) + \psi_1(x) = \phi(x), && c_1\varphi_1(x) - c_1\psi_1(x) = \phi'(x), && x>0, && (3.8)\\
&\varphi_2(x) + \psi_2(x) = 0, && c_2\varphi_2(x) - c_2\psi_2(x) = 0 && x<0. && (3.9)
\end{aligned}$$

which (up to constants which do not affect $u(x,t)$) imply

$$\begin{aligned}
&\varphi_1(x) = \phi(x), && \psi_1(x) = 0, && x>0, && (3.10)\\
&\varphi_2(x) = 0, && \psi_2(x) = 0 && x<0. && (3.11)
\end{aligned}$$

which gives us $u(x,t) = \phi(x + c_1t)$ as $x > c_1t$ and $u(x,t) = 0$ as $x_1 < -c_2t$. Now we are doing the most important thing: finding $\psi_1(x)$ as $x < 0$ and $\varphi_2(x)$ as $x > 0$. From (7.5) as $t > 0$:

$$\begin{aligned}
\varphi_1(c_1t) + \psi_1(-c_1t) &= \alpha\bigl(\varphi_2(c_2t) + \psi_2(-c_2t)\bigr),\\
\varphi_1'(c_1t) + \psi_1'(-c_1t) &= \beta\bigl(\varphi_2'(c_2t) + \psi_2'(-c_2t)\bigr)
\end{aligned}$$

and plugging (4.11) and (4.12) we get

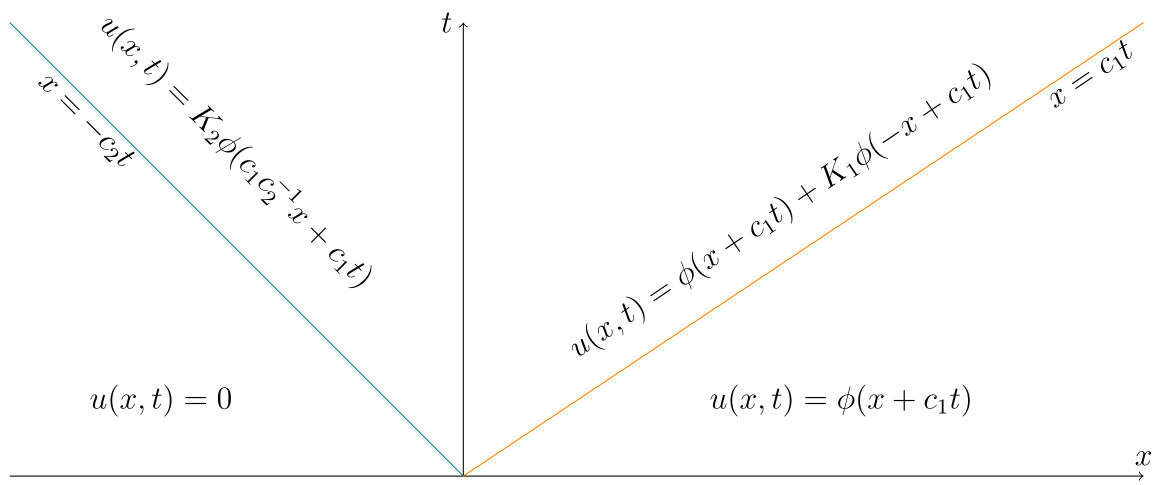
$$\phi(c_1t) + \psi_1(-c_1t) = \alpha\varphi_2(c_2t), \tag{3.12}$$
$$c_1^{-1}\phi(c_1t) - c_1^{-1}\psi_1(c_1t) = \beta c_2^{-1}\varphi(c_2t) \tag{3.13}$$

where we already integrated $\phi'(c_1t) + \psi_1'(-c_1t) = \beta\varphi_2'(c_2t)$. Solving this system we get

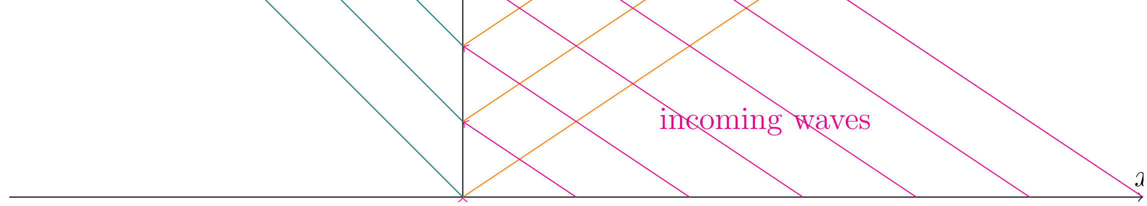
$$\begin{aligned}
&\psi_1(-c_1t) = K_1\phi(c_1t), && K_1 = (\alpha c_1^{-1} - \beta c_2^{-1})/(\alpha c_1^{-1} + \beta c_2^{-1}),\\
&\varphi_2(c_2t) = K_2\phi(c_1t), && K_2 = 2c_1^{-1}/(\alpha c_1^{-1} + \beta c_2^{-1})
\end{aligned}$$

which implies what on the figure

$\square$



t
x
$u(x,t) = K_2\phi(c_1 c_2^{-1} x + c_1 t)$
$x = -c_2 t$
$u(x,t) = \phi(x + c_1 t) + K_1\phi(-x + c_1 t)$
$x = c_1 t$
$u(x,t) = 0$
$u(x,t) = \phi(x + c_1 t)$



t
refracted waves
reflected waves
incoming waves

4 Quiz 4

Problem 4.1 (5pt). Oscillations of the beam are described by equation

$$u_{tt} + K u_{xxxx} = 0, \qquad 0 < x < l. \tag{4.1}$$

with $K > 0$.
If both ends clamped (that means having the fixed positions and directions) then the boundary conditions are

$$u(0,t) = u_x(0,t) = 0, \tag{4.2}$$
$$u(l,t) = u_x(l,t) = 0. \tag{4.3}$$

(a) Find equation describing frequencies and corresponding eigenfunctions (You may assume that all eigenvalues are real and positive).

(d) **Bonus** [+1pt] Prove that eigenvalues are simple, i.e. all eigenfunctions corresponding to the same eigenvalue are proportional.

Hint. Change coordinate system so that interval becomes [-L,L] with L=l/2; consider separately even and odd eigenfunctions.

Solution. Separating variables $u(x,t) = X(x)T(t)$ we arrive to

$$X^{IV} = \omega^4 X, \tag{4.4}$$
$$X(-L) = X'(-L) = 0, \tag{4.5}$$
$$X(L) = X'(L) = 0 \tag{4.6}$$

and

$$T'' + K\omega^4 T = 0 \tag{4.7}$$

with $\omega > 0$ (see Hint).

(a) Solving characteristic equation $k^4 = \omega^4$ we get $k_{1,2} = \pm\omega$, $k_{3,4} = \pm i\omega$ and

$$X = A\cosh(\omega x) + B\sinh(\omega x) + C\cos(\omega x) + D\sin(\omega x). \tag{4.8}$$

Plugging into (7.5), (6.6) we get (dividing $X'(\pm L)$ by ω

$$A\cosh(\omega L) + B\sinh(\omega L) + C\cos(\omega L) + D\sin(\omega L) = 0, \tag{4.9}$$
$$A\sinh(\omega L) + B\cosh(\omega L) - C\sin(\omega L) + D\cos(\omega L) = 0, \tag{4.10}$$
$$A\cosh(\omega L) - B\sinh(\omega L) + C\cos(\omega L) - D\sin(\omega L) = 0, \tag{4.11}$$
$$-A\sinh(\omega L) + B\cosh(\omega L) + C\sin(\omega L) + D\cos(\omega L) = 0 \tag{4.12}$$

and immediately

$$A\cosh(\omega L) + C\cos(\omega L) = 0 \tag{4.13}$$
$$A\sinh(\omega L) - C\sin(\omega L) = 0, \tag{4.14}$$

and

$$B\sinh(\omega L) + D\sin(\omega L) = 0, \tag{4.15}$$
$$B\cosh(\omega L) + D\cos(\omega L) = 0 \tag{4.16}$$

The first system has non-trivial solution iff its determinant is 0

$$\cosh(\omega L)\sin(\omega L) + \cos(\omega L)\sinh(\omega L) = 0 \iff \tanh(\omega L) = -\tan(\omega L). \tag{4.17}$$

The second system has non-trivial solution iff its determinant is 0

$$\cosh(\omega L)\sin(\omega L) - \cos(\omega L)\sinh(\omega L) = 0 \iff \tanh(\omega L) = \tan(\omega L). \tag{4.18}$$

so ω must satisfy either (4.17) or (4.18).

Then, in case (4.17) $B = D = 0$, and up to a constant factor $C = 1$, $A = -\cos(\omega L)/\cosh(\omega L)$ and

$$X(x) = \cos(\omega x) - \frac{\cosh(\omega x)\cos(\omega L)}{\cosh(\omega L)}. \tag{4.19}$$

Similarly, in case (4.17)

$$X(x) = \sin(\omega x) - \frac{\sinh(\omega x)\sin(\omega L)}{\sinh(\omega L)}. \tag{4.20}$$

(d) (4.17) and (4.18) are not compatible. Indeed, if both hold then $\tanh(\omega L) = 0$ which contradicts to $\omega L > 0$.

Also if (4.17) or (4.18) holds then the corresponding matrix has rank 1: indeed $\cosh(\omega L) \neq 0$.

Therefore the space of solution is 1-dimensional.

$\square$

Solution 2. Not following Hint. Still

$$X = A\cosh(\omega x) + B\sinh(\omega x) + C\cos(\omega x) + D\sin(\omega x). \tag{4.21}$$

and as $x = 0$: $A + C = 0$, $B + D = 0$ and

$$X = A\bigl(\cosh(\omega x) - \cos(\omega x)\bigr) + B\bigl(\sinh(\omega x) - \sin(\omega x)\bigr). \tag{4.22}$$

Then as $x = l$

$$A\bigl(\cosh(\omega l) - \cos(\omega l)\bigr) + B\bigl(\sinh(\omega l) - \sin(\omega l)\bigr) = 0,$$
$$A\bigl(\sinh(\omega l) + \sin(\omega l)\bigr) + B\bigl(\cosh(\omega l) - \cos(\omega l)\bigr) = 0$$

where we divided $X'(l)$ by ω. Then it has non-trivial solution if its determinant is 0:

$$\big(\cosh(\omega l)-\cos(\omega l)\big)^2-\big(\sinh(\omega l)+\sin(\omega l)\big)\big(\sinh(\omega l)-\sin(\omega l)\big)=0$$

which rewritten as

$$\cosh^2(\omega l)-2\cosh(\omega l)\cos(\omega l)+\cos^2(\omega l)-\sinh^2(\omega l)+\sin^2(\omega l)=0$$

and since $\cosh^2(t)-\sinh^2(t)=1$, $\cos^2(t)+\sin^2(t)=1$ we get

$$\cosh(\omega l)\cos(\omega l)=1 \tag{4.23}$$

and taking $B=\big(\cosh(\omega l)-\cos(\omega l)\big)$, $A=-\big(\sinh(\omega l)-\sin(\omega l)\big)$ (up to a common factor) we get

$$\begin{aligned}X=-\big(\sinh(\omega l)-\sin(\omega l)\big)\big(\cosh(\omega x)-\cos(\omega x)\big)+\\ \big(\cosh(\omega l)-\cos(\omega l)\big)\big(\sinh(\omega x)-\sin(\omega x)\big).\end{aligned} \tag{4.24}$$

It is the same solution as Solution 1: one can prove that

$$\cosh(2\omega L)\cos(2\omega L)=1 \iff \tanh^2(\omega L)=\tan^2(\omega L)$$

(remember, $l=2L$), and X differ by a shift (by $-L$) and a factor. $\square$

5 Quiz 5

Problem 5.1 (5pts)**.** As $\alpha>0$ find Fourier transforms of

(a) (2pts) $e^{-|x|}$;

(b) (1.5pts) $e^{-|x|}\sin(x)$;

(c) (1.5pts) $xe^{-|x|}\sin(x)$.

Solution. (a) Fourier transform of $e^{-|x|}$:

$$\begin{aligned}(2\pi)^{-1}\int_{-\infty}^{\infty}e^{-|x|-ikx}\,dx=(2\pi)^{-1}\Big(\int_{-\infty}^{0}e^{(1-ik)x}\,dx+\int_0^{\infty}e^{(-1-ik)x}\,dx\Big)=\\ (2\pi)^{-1}\Big((1-ik)^{-1}e^{(1-ik)x}\Big|_{x=-\infty}^{x=0}+(-1-ik)e^{(-1-ik)x}\Big|_{x=0}^{x=\infty}\Big)=\\ (2\pi)^{-1}\Big((1-ik)^{-1}-(-1-ik)^{-1}\Big)=\frac{1}{\pi(1+k^2)}.\end{aligned} \tag{5.1}$$

(b) Since $\sin(x)=\frac{1}{2i}\Big(e^{ix}-e^{-ix}\Big)$ and multiplication by $e^{i\beta x}$ of u means $k\mapsto k-\beta$ for $\hat{u}$, F.T. of $e^{-|x|}\sin(x)$ is

$$\frac{1}{2i\pi}\Big(\frac{1}{1+(k-1)^2}-\frac{1}{1+(k+1)^2}\Big). \tag{5.2}$$

(c) Multiplication of u by x means $i\partial_k \hat{u}(k)$ resulting in

$$\frac{1}{2\pi}\partial_k\Bigl(\frac{1}{1+(k-1)^2}-\frac{1}{1+(k+1)^2}\Bigr)=\\ -\frac{1}{2\pi}\biggl(\frac{k-1}{\bigl(1+(k-1)^2\bigr)^2}-\frac{k+1}{\bigl(1+(k+1)^2\bigr)^2}\biggr). \tag{5.3}$$

□

6 Quiz 6

Problem 6.1 (5pt)**.** Solve

$$\begin{aligned} &\Delta u := u_{xx}+u_{yy}=0 && \text{in } \ r<a\\ &u|_{r=a}=f(\theta). \end{aligned}$$

where we use polar coordinates (r,θ) and $f(\theta)=\begin{cases} 1 & 0<\theta<\pi\\ -1 & \pi<\theta<2\pi.\end{cases}$

Hint. Use Fourier method rather than Poisson formula.

Solution. Setting $u(r,\theta)=R(r)\Theta(\theta)$ we after separation of variables arrive to

$$\frac{r^2R''+rR'}{R}+\frac{\Theta''}{\Theta}=0$$

and since the first term depends on r only, and the second term on θ only we conclude that both are constant; also Θ must be 2π-periodic and therefore

$$\Theta''+\lambda\Theta=0, \qquad \Theta(\theta+2\pi)=\Theta(\theta); \tag{6.1}$$

then $\lambda_0=0$, $\Theta_0=\frac{1}{2}$ and $\lambda_n=\pi^2n^2$, $\Theta_{n,1}=\cos(n\theta)$, $\Theta_{n,2}=\sin(n\theta)$ for $n=1,2,\ldots$.
Then

$$r^2R''+rR'+n^2R=0. \tag{6.2}$$

This is Euler equation and its solutions are $R_0=A_0+B_0\ln r$,
$R_{n,i}=A_{n,i}r^n+B_{n,i}r^{-n}$ or $n=1,2,\ldots$.
We remove terms with $\ln r$ and r^{-n} since they are singular at the origin and finally

$$u(r,\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}\bigl(A_n\cos(n\theta)+B_n\sin(n\theta)\bigr)r^n. \tag{6.3}$$

Plugging into Dirichlet boundary condition we get

$$f(\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}\bigl(A_n\cos(n\theta)+B_n\sin(n\theta)\bigr)a^n. \tag{6.4}$$

Then

$$A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \qquad n = 0, 1, 2, \ldots$$
$$B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta \qquad n = 1, 2, \ldots$$

Since $f(\theta)$ is an odd function $A_n = 0$ and

$$B_n = \frac{2}{\pi a^n} \int_{0}^{\pi} f(\theta)\sin(n\theta)\,d\theta \qquad n = 1, 2, \ldots \tag{6.5}$$

Note No punishment for using all these formulate without deduction.
Then

$$B_n = \frac{2}{\pi a^n} \int_0^{\pi} \sin(n\theta)\,d\theta = -\frac{2}{n\pi a^2} \cos(n\theta)\Big|_{\theta=0}^{\theta=\pi} = \begin{cases} 0 & n = 2m, \\ \dfrac{4}{(2m+1)\pi a^2} & n = 2m+1. \end{cases} \tag{6.6}$$

Finally

$$u(r,\theta) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi a^2} \sin(2m+1)\theta.$$

□

7 Quiz 7

Problem 7.1. The heavy flexible but unstretchable wire (chain) has a length and an energy respectively

$$\ell = \int_{-1}^{1} \sqrt{1 + u'^2}\,dx, \tag{7.1}$$
$$U = \rho g \int_{-1}^{1} u\sqrt{1 + u'^2}\,dx \tag{7.2}$$

where ρ is a linear density.

(a) Write down an equation minimizing energy U as length $\ell = 4$ is fixed.

(b) Find solution satisfying $u(-1) = u(1) = 0$.

(c) (bonus) Calculate U.

Hint Since Lagrangian L does not depend on x explicitly, Euler-Lagrange equation is equivalent to $H := u'L_{u'} - L = \text{const}$.

Solution. (a) Euler-Lagrange functional is

$$\int_{-1}^{1} (u-\lambda)\sqrt{1+u'^2}\, dx \tag{7.3}$$

with Lagrangian

$$L = (u-\lambda)\sqrt{1+u'^2}. \tag{7.4}$$

Constant factor ρg does not matter here. Then equation is

$$\sqrt{1+u'^2} - \Bigl(\frac{(u-\lambda)u'}{\sqrt{1+u'^2}}\Bigr)' = 0. \tag{7.5}$$

(b) Simpler to use Hint than to solve equation. Then

$$H = -\frac{(u-\lambda)}{\sqrt{1+u'^2}} = \text{const} \implies$$

$$\sqrt{1+u'^2} = A(y-\lambda) \implies \frac{du}{A^2(u-\lambda)^2-1} = dx \tag{7.6}$$

and integration gives the answer

$$u = \frac{1}{A}\cosh(A(x-B)) + \lambda. \tag{7.7}$$

We have three parameters and three equations. $u(-1) = u(1) = 0 \implies B = 0$, $\lambda = -\frac{1}{A}\cosh(A)$. So,

$$u = \frac{1}{A}\cosh(Ax) - \frac{1}{A}\cosh(A) \tag{7.8}$$

where A is a root of

$$\int_{-1}^{1} \sqrt{1+u'^2}\, dx = \int_{-1}^{1} \cosh(Ax)\, dx = \frac{2}{A}\sinh(A) = \ell$$

i.e. $\sinh(A) = 2A$.

(c) Assuming for simplicity $\rho g = 1$

$$\begin{aligned} U = \int_{-1}^{1}\int_{-1}^{1} u\sqrt{1+u'^2} &= \frac{1}{A}\int_{-1}^{1} \bigl(\cosh(Ax) - \cosh(A)\bigr)\cosh(Ax) = \\ &\frac{1}{A}\int_{-1}^{1} \Bigl(\frac{1}{2}\cosh(2Ax) + \frac{1}{2} - \cosh(Ax)\cosh(A)\Bigr)\, dx = \\ &\frac{1}{A^2}\Bigl(\frac{1}{2}\sinh(2A) + A - 2\sinh(A)\cosh(A)\Bigr) = \\ &\frac{1}{A^2}\Bigl(A - \sinh(A)\cosh(A)\Bigr) = \frac{1}{A}\Bigl(1 - 2\cosh(A)\Bigr). \end{aligned} \tag{7.9}$$

Numerics show that $A \approx 2.17732 \implies U \approx -3.64484$ (not required).

□