Deadline Monday, September 24, 9 pm

APM 346 (2012) Home Assignment 1

Problem 1

a. Find general solution

$$u_x - 3u_y = 0; \tag{1}$$

b. Solve IVP problem $u|_{x=0} = e^{-y^2}$ for equation (1) in $\mathbb{R}^2$;

c. Consider equation (1) in $\{x > 0, y > 0\}$ with the initial condition $u|_{x=0} = y$ ($y > 0$); where this solution defined? Is it defined everywhere in $\{x > 0, y > 0\}$ or do we need to impose condition at $y = 0$? In the latter case impose condition $u|_{y=0} = x$ ($x > 0$) and solve this IVBP;

d. Consider equation (1) in $\{x < 0, y > 0\}$ with the initial condition $u|_{x=0} = y$ ($y > 0$); where this solution defined? Is it defined everywhere in $\{x < 0, y > 0\}$ or do we need to impose condition at $y = 0$? In the latter case impose condition $u|_{y=0} = x$ ($x < 0$) and solve this IVBP.

Problem 2

Corrected

a. Find the general solution of

$$xu_x + 4yu_y = 0 \tag{2}$$

in $\{(x, y) \neq (0, 0)\}$; when this solution is continuous at $(0, 0)$?

b. Find the general solution of

$$xu_x - 4yu_y = 0 \tag{3}$$

in $\{(x, y) \neq (0, 0)\}$; when this solution is continuous at $(0, 0)$?

c. Explain the difference.

Problem 3

Find the solution of

$$\begin{cases} u_x + 3u_y = xy, \\ u|_{x=0} = 0. \end{cases} \tag{4}$$

Problem 4

Corrected

a. Find the general solution of

$$yu_x - xu_y = xy; \tag{5}$$

b. Find the general solution of

$$yu_x - xu_y = x^2 + y^2; \tag{6}$$

c. In one instanse solution does not exist. Explain why.

Problem 5

a. Find the general solution of

$$u_{tt} - 9u_{xx} = 0; \tag{7}$$

b. Solve IVP

$$u|_{t=0} = x^2, \quad u_t|_{t=0} = x \tag{8}$$

for (7);

c. Consider (7) in $\{x > 3t, x > -3t\}$ and find a solution to it, satisfying Goursat problemCorrected September 23

$$u|_{x=3t} = t, \quad u|_{x=-3t} = 2t. \tag{9}$$

Problem 6

Derivation of a PDE describing traffic flow. The purpose of this problem is to derive a model PDE that describes a congested one-dimensional highway. Let

- $\rho(x,t)$ denote the traffic density : the number of cars per kilometer at time t located at position x;
- $q(x,t)$ denote the traffic flow: the number of cars per hour passing a fixed

place x at time t;

- $N(t, a, b)$ denote the number of cars between position $x = a$ and $x = b$ at time t.

a. Derive a formual for $N(t, a, b)$ as an integral of the traffic density. You can assume there are no exits or entrances between position a and b.
b. Derive a formula for $\frac{\partial N}{\partial t}$ depending on the traffic flow. '''Hint:''' You can express the change in cars between time $t_1 = t$ and $t_2 = t + h$ in terms of of traffic flow;
c. Differentiate with respect to t the integral form for N from part (a) and make it equal to the formula you got in part (b). This of the integral form of conservation of cars;
d. Express the right hand side of the formula of part (c) in terms of an integral. Since a, b are arbitrary, obtain a PDE. This PDE is called the conservation of cars equation;
e. What equation do you get in part (4) if $q = c\rho$, for some constant c. What choice of c would be more realistic, i.e. what should c be function of?