

#5 Diffusion Equation (1) with $\phi(x) = e^{-x}$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) e^{-y} dy$$

$$= \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-\frac{(x-y)^2 + 4kty}{4kt}} dy}_{(1)} - \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-\frac{(x+y)^2 + 4kty}{4kt}} dy}_{(2)}$$

First part: (By using formula $(a+b+c)^2 = a^2 + b^2 + c^2 + 2cab + bc + ac$)

$$(1) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-\left(\frac{-x+y+2kt}{\sqrt{4kt}}\right)^2 + kt - x} dy$$

$$\text{Let } p = \frac{-x+y+2kt}{\sqrt{4kt}} \quad \frac{\partial p}{\partial y} = \frac{1}{\sqrt{4kt}} \quad p|_{y=0} = \frac{-x+2kt}{\sqrt{4kt}}$$

$$(1) = \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{\frac{-x+2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2} dp$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-p^2} dp - \int_0^{\frac{2kt-x}{\sqrt{4kt}}} e^{-p^2} dp \right)$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{Erf} \left(\frac{2kt-x}{\sqrt{4kt}} \right) \right)$$

$$= e^{kt-x} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \operatorname{Erf} \left(\frac{2kt-x}{\sqrt{4kt}} \right) \right)$$

Second Part: (2) = $\frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-\left(\frac{x+y+2kt}{\sqrt{4kt}}\right)^2 + x + kt} dy$

$$\text{Let } p = \frac{x+y+2kt}{\sqrt{4kt}} \quad \frac{\partial p}{\partial y} = \frac{1}{\sqrt{4kt}} \quad p|_{y=0} = \frac{x+2kt}{\sqrt{4kt}}$$

$$(2) = \frac{e^{x+kt}}{\sqrt{\pi}} \left(\int_{\frac{x+2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2} dp \right)$$

$$= \frac{e^{x+kt}}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-p^2} dp - \int_0^{\frac{x+2kt}{\sqrt{4kt}}} e^{-p^2} dp \right)$$

$$= e^{x+kt} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \operatorname{Erf} \left(\frac{x+2kt}{\sqrt{4kt}} \right) \right)$$

$k=9$.

$$u(x,t) = (1) - (2) = e^{9t-x} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \operatorname{Erf} \left(\frac{18t-x}{\sqrt{36t}} \right) \right) + e^{x+9t} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \operatorname{Erf} \left(\frac{x+18t}{\sqrt{36t}} \right) \right)$$