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Section ....

#	points	Mark
1	[20]	
2	[20]	
3	[20]	
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5	[20]	
Total	[100]	

1. (20 pts) Consider the first order equation:

$$u_t + xu_x = 0. \tag{1}$$

(a) Find the characteristic curves and sketch them in the (x, t) plane.

(b) Write the general solution.

(c) Solve equation (1) with the initial condition $u(x, 0) = \cos(2x)$. Explain why the solution is fully determined by the initial condition.

(d) **(bonus 5 pts)** Describe domain in which solution of

$$u_t + x^2 u_x = 0, \qquad x > 0 \tag{2}$$

is fully determined by the initial condition $u(x, 0) = g(x)$ $(x > 0)$.

Continued

2. (20 pts) Consider the initial value problem for the wave equation posed on the left half-line:

$$\begin{cases} u_{tt} - u_{xx} = 0, & -\infty < x < 0 \\ u(x,0) = f(x), & -\infty < x < 0, \\ u_t(x,0) = g(x), & -\infty < x < 0. \end{cases}$$

Do the initial conditions uniquely determine the solution in the region $\{(t,x) : t \in \mathbb{R}, -\infty < x < 0\}$. Explain your answer with convincing arguments.

3. (20 pts) Consider the PDE with boundary conditions:

$$\begin{aligned} &u_{tt} + K u_{xxxx} + \omega^2 u = 0, && 0 < x < L, \\ &u(0,t) = u_x(0,t) = 0, && \\ &u(L,t) = u_x(L,t) = 0, && \end{aligned}$$

where $K > 0$ is constant. Prove that the energy $E(t)$ defined as

$$E(t) = \frac{1}{2} \int_0^L \left(u_t^2 + K u_{xx}^2 + \omega^2 u^2 \right) dx$$

does not depend on t.

Continued

4. (20 pts) Consider the initial value problem for the diffusion equation on the line:

$$\begin{cases} u_t = k u_{xx}, & x \in \mathbb{R}, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

(a) Assuming f is smooth and vanishes when $|x| > 10$, give a formula for the solution $u(x,t)$.

(b) Using the formula from part (a), prove that

$$\lim_{t \searrow 0} u(x,t) = f(x).$$

5. (20 pts) The functions H and Q are defined as follows:

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

$$Q(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Consider the function

$$C(x) = \int_{-\infty}^{+\infty} H(x-y)Q(y)dy.$$

(a) Graph the function $C(x)$.

(b) Identify the set of points x where $C(x) > 0$.

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